Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 584642, 10 pages
doi:10.1155/2010/584642

Research Article

A Note on $(\mathcal{C}_p, \alpha)$-Hyponormal Operators

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Received 20 January 2010; Accepted 22 April 2010

Academic Editor: Sin Takahasi

We study $(\mathcal{C}_p, \alpha)$-normal operators and $(\mathcal{C}_p, \alpha)$-hyponormal operators. We show the inclusion relation between these classes under various hypotheses for p and α. We also obtain some sufficient conditions for Aluthge transform $\widetilde{T}_{s,t} = |T|^s U |T|^t$ and T^2 of $(\mathcal{C}_p, \alpha)$-hyponormal operators still to be $(\mathcal{C}_p, \alpha)$-hyponormal.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. Recently, Lauric in [1] introduced $(\mathcal{C}_p, \alpha)$-hyponormal operators. For $\alpha > 0$ and $T \in \mathcal{L}(\mathcal{H})$, denote by $D_T^\alpha = (T^*T)^\alpha - (TT^*)^\alpha$. We denote that $\mathcal{C}_p(\mathcal{H})$, $1 \leq p < \infty$, the ideal of operators in the Schatten p-class [2]. Although, for $0 < p < 1$, the usual definition of $\|\cdot\|_p$ does not satisfy the triangle inequality, nevertheless $(\mathcal{C}_p, \|\cdot\|_p)$ is closed and $\|TK\|_p \leq \|T\| \cdot \|K\|_p$, when $T \in \mathcal{L}(\mathcal{H})$ and $K \in \mathcal{C}_p(\mathcal{H})$. An operator T in $\mathcal{L}(\mathcal{H})$ is $(\mathcal{C}_p, \alpha)$-normal if $D_T^\alpha \in \mathcal{C}_p(\mathcal{H})$, and denote the class of $(\mathcal{C}_p, \alpha)$-normal operators by $\mathcal{N}_p^\alpha(\mathcal{H})$. An operator T in $\mathcal{L}(\mathcal{H})$ will be called $(\mathcal{C}_p, \alpha)$-hyponormal if $D_T^\alpha = P + K$, where P is a positive semidefinite operator ($P \geq 0$) and $K \in \mathcal{C}_p(\mathcal{H})$. The class of $(\mathcal{C}_p, \alpha)$-hyponormal operators will be denoted by $\mathcal{H}_p^\alpha(\mathcal{H})$. In particular, an operator T in $\mathcal{H}_1^1(\mathcal{H})$ will be called almost hyponormal. Furthermore, an operator $T \in \mathcal{L}(\mathcal{H})$ whose D_T^α is positive semidefinite is called α-hyponormal (notation: $T \in \mathcal{H}_0^\alpha(\mathcal{H})$).

In this paper, we first study the inclusion relation between these classes under various hypotheses for p and α in Section 2. Then we study the Aluthge transform $\widetilde{T}_{s,t} = |T|^s U |T|^t$ and T^2 of $(\mathcal{C}_p, \alpha)$-hyponormal operators in Section 3.

Before proceeding, we will make use of the following inequality.

Theorem F (See Furuta inequality in [3]). *If $A \geq B \geq 0$, then, for each $r \geq 0$,*

$$\left(B^{r/2}A^pB^{r/2}\right)^{1/q} \geq \left(B^{r/2}B^pB^{r/2}\right)^{1/q},$$
$$\left(A^{r/2}A^pA^{r/2}\right)^{1/q} \geq \left(A^{r/2}B^pA^{r/2}\right)^{1/q}, \tag{1.1}$$

as long as real numbers p, r, q satisfy

$$p \geq 0, q \geq 1 \text{ with } (1+r)q \geq p+r. \tag{1.2}$$

Lemma 1.1 (see [1]). *Let $A \in \mathcal{L}(\mathcal{H})$, $A \geq 0$, $\alpha \in (0,1]$, $p \geq \alpha$, and $K \in \mathcal{C}_p(\mathcal{H})$, such that $A+K \geq 0$. Then $(A+K)^\alpha = A^\alpha + K_1$, where $K_1 \in \mathcal{C}_{p/\alpha}(\mathcal{H})$. If in addition $K \geq 0$, then $K_1 \geq 0$.*

Lemma 1.2 (see [1]). *Let $A \in \mathcal{L}(\mathcal{H})$, $A \geq 0$, $p \geq 1$, and $K \in \mathcal{C}_p(\mathcal{H})$, such that $A+K \geq 0$, and let $\alpha \in [1,+\infty)$. Then $(A+K)^\alpha = A^\alpha + K_1$, where $K_1 \in \mathcal{C}_p(\mathcal{H})$.*

2. Some Inclusions

According to Löwner-Heinz (L-H) inequality [4, 5] that $A \geq B \geq 0$ ensures that $A^\alpha \geq B^\alpha$ for each $\alpha \in [0,1]$, we obtain $\mathcal{H}_0^\alpha(\mathcal{H}) \supseteq \mathcal{H}_0^\beta(\mathcal{H})$ when $\alpha \leq \beta$. However, the similar inclusions for the classes $\mathcal{N}_p^\alpha(\mathcal{H})$ and $\mathcal{H}_p^\alpha(\mathcal{H})$ are less obvious. In this section, we will examine various inclusions between these classes of operators. (1) of Theorem 2.1 has been already shown in [1]. But we will give a proof for the readers' convenience.

Theorem 2.1. *Let $\alpha > 0$, $p \geq 1$, and let T be in $\mathcal{N}_p^\alpha(\mathcal{H})$.*

(1) *If $\beta \geq \alpha$, then T belongs to $\mathcal{N}_p^\beta(\mathcal{H})$, and therefore $\mathcal{N}_p^\alpha(\mathcal{H}) \subseteq \mathcal{N}_p^\beta(\mathcal{H})$.*

(2) *If $0 < \beta \leq \alpha$, then T belongs to $\mathcal{N}_{\alpha p/\beta}^\beta(\mathcal{H})$, and therefore $\mathcal{N}_p^\alpha(\mathcal{H}) \subseteq \mathcal{N}_{\alpha p/\beta}^\beta(\mathcal{H})$.*

Proof. Let α, p, and T be as in the hypotheses and let $T = U|T|$ be the polar decomposition of T.

For $T \in \mathcal{N}_p^\alpha(\mathcal{H})$, we have

$$D_T^\alpha = (T^*T)^\alpha - (TT^*)^\alpha = |T|^{2\alpha} - |T^*|^{2\alpha} = K, \tag{2.1}$$

with $K \in \mathcal{C}_p(\mathcal{H})$. Then we obtain

$$|T|^{2\alpha} = |T^*|^{2\alpha} + K \geq 0. \tag{2.2}$$

(1) First we consider the case $\beta \geq \alpha$. According to Lemma 1.2, we obtain

$$|T|^{2\beta} = \left(|T^*|^{2\alpha} + K\right)^{\beta/\alpha} = |T^*|^{2\beta} + K_1, \tag{2.3}$$

with $K_1 \in \mathcal{C}_p(\mathcal{H})$. Then $T \in \mathcal{N}_p^\beta(\mathcal{H})$.

(2) Next we consider the case $0 < \beta \le \alpha$. According to Lemma 1.1, we obtain

$$|T|^{2\beta} = \left(|T^*|^{2\alpha} + K\right)^{\beta/\alpha} = |T^*|^{2\beta} + K_1, \tag{2.4}$$

with $K_1 \in \mathcal{C}_{\alpha p/\beta}(\mathcal{H})$. Then $T \in \mathcal{N}^{\beta}_{\alpha p/\beta}(\mathcal{H})$. □

The following corollary is a consequence of Theorem 2.1.

Corollary 2.2. *Let $\alpha > 0$, $p \ge 1$, then, for $0 < \beta \le \alpha$,*

$$\mathcal{N}^{\beta}_{p}(\mathcal{H}) \subseteq \mathcal{N}^{\alpha}_{p}(\mathcal{H}) \subseteq \mathcal{N}^{\beta}_{\alpha p/\beta}(\mathcal{H}) \subseteq \mathcal{N}^{\alpha}_{\alpha p/\beta}(\mathcal{H}). \tag{2.5}$$

Theorem 2.3. *Let $\alpha > 0$, $p \ge 1$, and let T be in $\mathcal{H}^{\alpha}_{p}(\mathcal{H})$. If $0 < \beta \le \alpha$, then T belongs to $\mathcal{H}^{\beta}_{\alpha p/\beta}(\mathcal{H})$, and therefore $\mathcal{H}^{\alpha}_{p}(\mathcal{H}) \subseteq \mathcal{H}^{\beta}_{\alpha p/\beta}(\mathcal{H})$.*

Proof. Let α, p, and T be as in the hypotheses and let $T = U|T|$ be the polar decomposition of T.

For $T \in \mathcal{H}^{\alpha}_{p}(\mathcal{H})$, we have

$$D^{\alpha}_{T} = (T^*T)^{\alpha} - (TT^*)^{\alpha} = |T|^{2\alpha} - |T^*|^{2\alpha} = P + K, \tag{2.6}$$

with $P \ge 0$, $K \in \mathcal{C}_p(\mathcal{H})$. Then we obtain

$$|T|^{2\alpha} = |T^*|^{2\alpha} + P + K \ge 0. \tag{2.7}$$

For $0 < \beta \le \alpha$, according to Lemma 1.1 and L-H inequality, we obtain

$$\begin{aligned} |T|^{2\beta} &= \left(|T^*|^{2\alpha} + P + K\right)^{\beta/\alpha} \\ &= \left(|T^*|^{2\alpha} + P\right)^{\beta/\alpha} + K_1 \\ &\ge |T^*|^{2\beta} + K_1, \end{aligned} \tag{2.8}$$

with $K_1 \in \mathcal{C}_{\alpha p/\beta}(\mathcal{H})$. Then we obtain $T \in \mathcal{H}^{\beta}_{\alpha p/\beta}(\mathcal{H})$. □

3. Some Properties of $(\mathcal{C}_p, \alpha)$-Hyponormal Operators

Let $T = U|T|$ be the polar decomposition of an operator T on a Hilbert space $\mathcal{H}$, where U is a partial isometry operator. Recently, Lauric [1] shows some theorems on the Aluthge transform $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ of $(\mathcal{C}_p, \alpha)$-hyponormal operators. In this section, we will show some sufficient conditions for the generalized Aluthge transform $\widetilde{T}_{s,t} = |T|^{s}U|T|^{t}\,(s, t > 0)$ and

T^2 of $(\mathcal{C}_p, \alpha)$-hyponormal operators to be $(\mathcal{C}_p, \alpha)$-hyponormal. Aluthge transform $\widetilde{T}_{s,t}$ arose in the study of p-hyponormal operators [6, 7] and has since been studied in many different contexts [8–15].

Let T belong to $\mathscr{H}_p^{\alpha}(\mathscr{H})$, for some $\alpha > 0, p > 0$, such that $D_T^{\alpha} = P + K$ with $P \geq 0$, $K \in \mathcal{C}_p(\mathscr{H})$. Since $K = K^* = K_+ - K_-$ and $K_+, K_- \geq 0$ are $\mathcal{C}_p$-class operators, in what follows we will assume that $D_T^{\alpha} = P_1 - K_1$ with $P_1 \geq 0$ and $K_1 \geq 0, K_1 \in \mathcal{C}_p(\mathscr{H})$.

Theorem 3.1. *Let $p \geq 1$, $\alpha \geq \max\{s, t\}$, and $T \in \mathscr{H}_p^{\alpha}(\mathscr{H})$ such that $D_T^{\alpha} = P - K$ with $P, K \geq 0$, $K \in \mathcal{C}_p(\mathscr{H})$, and let $\varepsilon \in (0, 1/2]$ such that $\alpha + \varepsilon \leq 1$. Then $\widetilde{T}_{s,t} \in \mathscr{H}_{2\alpha p/(\alpha+\varepsilon)s}^{(\alpha+\varepsilon)}(\mathscr{H})$.*

Proof. We may assume that $T = U|T|$ with U being unitary. The equality $D_T^{\alpha} = P - K$ with P, $K \geq 0$ implies that $|T|^{2\alpha} + K \geq U|T|^{2\alpha}U^*$. Multiplying this inequality by U^* to the left and by U to the right, we obtain

$$A = U^*|T|^{2\alpha}U + U^*KU \geq |T|^{2\alpha} = B. \tag{3.1}$$

According to Lemma 1.1,

$$A^{s/\alpha} = \left\{U^*\left(|T|^{2\alpha} + K\right)U\right\}^{s/\alpha} = U^*\left(|T|^{2\alpha} + K\right)^{s/\alpha}U = U^*\left(|T|^{2s} + K_1\right)U, \tag{3.2}$$

with $K_1 \in \mathcal{C}_{\alpha p/s}(\mathscr{H})$. Setting $K_2 = |T|^t U^* K_1 U |T|^t$, by Theorem F we have

$$\begin{aligned}
\left(\widetilde{T}_{s,t}^*\widetilde{T}_{s,t} + K_2\right)^{\alpha+\varepsilon} &= \left\{|T|^t\left[U^*\left(|T|^{2s} + K_1\right)U\right]|T|^t\right\}^{\alpha+\varepsilon} \\
&= \left\{|T|^t\left[U^*\left(|T|^{2\alpha} + K\right)U\right]^{s/\alpha}|T|^t\right\}^{\alpha+\varepsilon} \\
&= \left(B^{t/2\alpha}A^{s/\alpha}B^{t/2\alpha}\right)^{\alpha+\varepsilon} \\
&\geq B^{(s+t)(\alpha+\varepsilon)/\alpha} \\
&= |T|^{2(s+t)(\alpha+\varepsilon)}.
\end{aligned} \tag{3.3}$$

On the other hand, according to Lemma 1.1,

$$\left(\widetilde{T}_{s,t}^*\widetilde{T}_{s,t} + K_2\right)^{\alpha+\varepsilon} = \left(\widetilde{T}_{s,t}^*\widetilde{T}_{s,t}\right)^{\alpha+\varepsilon} + K_3, \tag{3.4}$$

with $K_3 \in \mathcal{C}_{\alpha p/(\alpha+\varepsilon)s}(\mathscr{H})$. Then we have

$$\left(\widetilde{T}_{s,t}^*\widetilde{T}_{s,t}\right)^{\alpha+\varepsilon} + K_3 \geq |T|^{2(s+t)(\alpha+\varepsilon)}. \tag{3.5}$$

According to the following inequality

$$C = |T|^{2\alpha} + K \geq U|T|^{2\alpha}U^* = D, \tag{3.6}$$

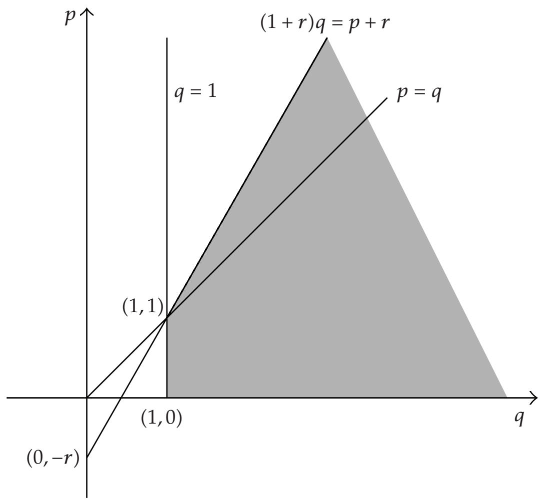

Figure 1: Domain of Furuta inequality.

by Theorem F, we have

$$\left(C^{s/2\alpha}D^{t/\alpha}C^{s/2\alpha}\right)^{\alpha+\varepsilon} \leq C^{(s+t)(\alpha+\varepsilon)/\alpha}. \tag{3.7}$$

Again, according to Lemma 1.1,

$$C^{s/2\alpha} = \left(|T|^{2\alpha} + K\right)^{s/2\alpha} = |T|^{s} + K_4, \tag{3.8}$$

with $K_4 \in \mathcal{C}_{2\alpha p/s}(\mathscr{H})$.

Next, obviously,

$$D^{t/\alpha} = \left(U|T|^{2\alpha}U^*\right)^{t/\alpha} = U|T|^{2t}U^*. \tag{3.9}$$

Then we have

$$\begin{aligned}
\left(C^{s/2\alpha}D^{t/\alpha}C^{s/2\alpha}\right)^{\alpha+\varepsilon} &= \left\{\left(|T|^{s} + K_4\right)\left(U|T|^{2t}U^*\right)\left(|T|^{s} + K_4\right)\right\}^{\alpha+\varepsilon} \\
&= \left(|T|^{s}U|T|^{2t}U^*|T|^{s} + K_5\right)^{\alpha+\varepsilon} \\
&= \left(\widetilde{T}_{s,t}\widetilde{T}^*_{s,t} + K_5\right)^{\alpha+\varepsilon} \\
&= \left(\widetilde{T}_{s,t}\widetilde{T}^*_{s,t}\right)^{\alpha+\varepsilon} + K_6,
\end{aligned} \tag{3.10}$$

with $K_5 \in \mathcal{C}_{2\alpha p/s}(\mathscr{H})$, $K_6 \in \mathcal{C}_{2\alpha p/(\alpha+\varepsilon)s}(\mathscr{H})$.

(1) First we consider the case $0 \leq ((s+t)/\alpha) \leq 1$. According to Lemma 1.1, we have

$$\begin{aligned} \left(C^{s+t/\alpha}\right)^{\alpha+\varepsilon} &= \left\{\left(|T|^{2\alpha}+K\right)^{s+t/\alpha}\right\}^{\alpha+\varepsilon} \\ &= \left(|T|^{2(s+t)}+K_7\right)^{\alpha+\varepsilon} \\ &= |T|^{2(s+t)(\alpha+\varepsilon)}+K_8, \end{aligned} \tag{3.11}$$

with $K_7 \in \mathcal{C}_{\alpha p/s+t}(\mathscr{H})$ and $K_8 \in \mathcal{C}_{\alpha p/(\alpha+\varepsilon)(s+t)}(\mathscr{H})$.

Then by (3.7) and (3.10), set $K_9 = K_6 - K_8 \in \mathcal{C}_{2\alpha p/(\alpha+\varepsilon)s}(\mathscr{H})$, and

$$|T|^{2(s+t)(\alpha+\varepsilon)} \geq \left(\widetilde{T}_{s,t}\widetilde{T}_{s,t}^{*}\right)^{\alpha+\varepsilon}+K_9. \tag{3.12}$$

Combining (3.5) and (3.12), we obtain

$$\left(\widetilde{T}_{s,t}^{*}\widetilde{T}_{s,t}\right)^{\alpha+\varepsilon}-\left(\widetilde{T}_{s,t}\widetilde{T}_{s,t}^{*}\right)^{\alpha+\varepsilon} \geq K_{10}, \tag{3.13}$$

where $K_{10} = K_9 - K_3 \in \mathcal{C}_{2\alpha p/(\alpha+\varepsilon)s}(\mathscr{H})$.

(2) Next we consider the case $(s+t/\alpha) > 1$. According to Lemmas 1.1 and 1.2,

$$\begin{aligned} \left(C^{s+t/\alpha}\right)^{\alpha+\varepsilon} &= \left\{\left(|T|^{2\alpha}+K\right)^{s+t/\alpha}\right\}^{\alpha+\varepsilon} \\ &= \left(|T|^{2(s+t)}+K_7'\right)^{\alpha+\varepsilon} \\ &= |T|^{2(s+t)(\alpha+\varepsilon)}+K_8', \end{aligned} \tag{3.14}$$

with $K_7' \in \mathcal{C}_{p}(\mathscr{H})$ and $K_8' \in \mathcal{C}_{p/\alpha+\varepsilon}(\mathscr{H})$. and

Then by (3.7) and (3.10), set $K_9' = K_6 - K_8' \in \mathcal{C}_{2\alpha p/(\alpha+\varepsilon)s}(\mathscr{H})$,

$$|T|^{2(s+t)(\alpha+\varepsilon)} \geq \left(\widetilde{T}_{s,t}\widetilde{T}_{s,t}^{*}\right)^{\alpha+\varepsilon}+K_9'. \tag{3.15}$$

Combining (3.5) and (3.15), we obtain

$$\left(\widetilde{T}_{s,t}^{*}\widetilde{T}_{s,t}\right)^{\alpha+\varepsilon}-\left(\widetilde{T}_{s,t}\widetilde{T}_{s,t}^{*}\right)^{\alpha+\varepsilon} \geq K_{10}', \tag{3.16}$$

where $K_{10}' = K_9' - K_3 \in \mathcal{C}_{2\alpha p/(\alpha+\varepsilon)s}(\mathscr{H})$.

By (3.13) and (3.16), we obtain $\widetilde{T}_{s,t} \in \mathscr{H}^{(\alpha+\varepsilon)}_{2\alpha p/(\alpha+\varepsilon)s}(\mathscr{H})$. □

Remark 3.2. The main theorem of [1] was considered in the case $\alpha \in [1/2, 1]$. Apparently, Theorem 3.1 implies (Theorems 13 in [1]) when $s = t = 1/2$. And we also obtain the following theorem.

Theorem 3.3. *Let $p \geq 1$, $0 < \alpha \leq \min\{s, t\}$, and $T \in \mathcal{H}_p^\alpha(\mathcal{H})$ such that $D_T^\alpha = P - K$ with $P, K \geq 0$, $K \in \mathcal{C}_p(\mathcal{H})$, and let $\varepsilon \geq 0$ such that $\alpha + \varepsilon \leq 2\alpha/(s+t)$.*

(1) *If $s \geq 2\alpha$, then $\widetilde{T}_{s,t} \in \mathcal{H}_{p/(\alpha+\varepsilon)}^{(\alpha+\varepsilon)}(\mathcal{H})$.*

(2) *If $0 < s < 2\alpha$, then $\widetilde{T}_{s,t} \in \mathcal{H}_{2\alpha p/(\alpha+\varepsilon)s}^{(\alpha+\varepsilon)}(\mathcal{H})$.*

Proof. The proof of Theorem 3.3 is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let $p \geq 1$, $T \in \mathcal{H}_p^\alpha(\mathcal{H})$ such that $D_T^\alpha = P - K$ with $P, K \geq 0$, $K \in \mathcal{C}_p(\mathcal{H})$, and let $\varepsilon \in (0, 1/2]$.*

(1) *If $\alpha \in (0, 1/4]$, then $\widetilde{T} \in \mathcal{H}_{p/(\alpha+\varepsilon)}^{(\alpha+\varepsilon)}(\mathcal{H})$.*

(2) *If $\alpha \in (1/4, 1/2]$, then $\widetilde{T} \in \mathcal{H}_{4\alpha p/(\alpha+\varepsilon)}^{(\alpha+\varepsilon)}(\mathcal{H})$.*

Proof. Put $s = t = 1/2$ in Theorem 3.3.

(1) When $\alpha \in (0, 1/4]$, we have $s \geq 2\alpha$. According to (1) of Theorem 3.3, then $\widetilde{T} \in \mathcal{H}_{p/(\alpha+\varepsilon)}^{(\alpha+\varepsilon)}(\mathcal{H})$.

(2) When $\alpha \in (1/4, 1/2]$, we have $0 < s < 2\alpha$. According to (2) of Theorem 3.3, then $\widetilde{T} \in \mathcal{H}_{4\alpha p/(\alpha+\varepsilon)}^{(\alpha+\varepsilon)}(\mathcal{H})$. □

Next, we will study T^2 of $(\mathcal{C}_p, \alpha)$-hyponormal operators. And first we will prove the following lemma.

Lemma 3.5. *Let $p \geq 1$, $\alpha \in (0, 1]$, and $T \in \mathcal{H}_p^\alpha(\mathcal{H})$ such that $D_T^\alpha = P + K$ with $P \geq 0$, $K \in \mathcal{C}_p(\mathcal{H})$, and $D_T^\alpha = P_1 - K_1$ with $P_1 \geq 0, K_1 \geq 0, K_1 \in \mathcal{C}_p(\mathcal{H})$. Then if $|T|^{2\alpha} - P \geq 0$, one has the following inequalities*

(1) *There exists $K' \in \mathcal{C}_{2p/\alpha}(\mathcal{H})$ such that $(|T||T^*|^2|T|)^{\alpha/2} + K' \leq |T|^{2\alpha}$.*

(2) *There exists $K'' \in \mathcal{C}_{2p/\alpha}(\mathcal{H})$ such that $(|T^*||T|^2|T^*|)^{\alpha/2} + K'' \geq |T^*|^{2\alpha}$.*

Proof. Let α, p, and T be as in the hypotheses and let $T = U|T|$ be the polar decomposition of T. Then we have

$$D_T^\alpha = (T^*T)^\alpha - (TT^*)^\alpha = |T|^{2\alpha} - |T^*|^{2\alpha} = P + K, \tag{3.17}$$

with $P \geq 0, K \in \mathcal{C}_p(\mathcal{H})$.

$$D_T^\alpha = |T|^{2\alpha} - |T^*|^{2\alpha} = P_1 - K_1, \tag{3.18}$$

with and $P_1 \geq 0, K_1 \geq 0,$ and $K_1 \in \mathcal{C}_p(\mathcal{H})$.

By (3.17), we have

$$A_1 = |T|^{2\alpha} \geq |T^*|^{2\alpha} + K = B_1 \geq 0. \tag{3.19}$$

And according to Lemma 1.2,

$$B_1^{1/\alpha} = \left(|T^*|^{2\alpha} + K\right)^{1/\alpha} = |T^*|^2 + K_2, \tag{3.20}$$

with $K_2 \in \mathcal{C}_p(\mathcal{H})$. Setting $K_3 = |T|K_2|T|$, by Theorem F we have

$$\begin{aligned}
\left(|T||T^*|^2|T| + K_3\right)^{\alpha/2} &= \left\{|T|\left(|T^*|^2 + K_2\right)|T|\right\}^{\alpha/2} \\
&= \left(A_1^{1/2\alpha} B_1^{1/\alpha} A_1^{1/2\alpha}\right)^{\alpha/2} \\
&\leq A_1 \\
&= |T|^{2\alpha}.
\end{aligned} \tag{3.21}$$

By (3.18), we have

$$A_2 = |T|^{2\alpha} + K_1 \geq |T^*|^{2\alpha} = B_2. \tag{3.22}$$

And according to Lemma 1.2,

$$A_2^{1/\alpha} = \left(|T|^{2\alpha} + K_1\right)^{1/\alpha} = |T|^2 + K_4, \tag{3.23}$$

with $K_4 \in \mathcal{C}_p(\mathcal{H})$. Setting $K_5 = |T^*|K_4|T^*|$, by Theorem F we have

$$\begin{aligned}
\left(|T^*||T|^2|T^*| + K_5\right)^{\alpha/2} &= \left\{|T^*|\left(|T|^2 + K_4\right)|T^*|\right\}^{\alpha/2} \\
&= \left(B_2^{1/2\alpha} A_2^{1/\alpha} B_2^{1/2\alpha}\right)^{\alpha/2} \\
&\geq B_2 \\
&= |T^*|^{2\alpha}.
\end{aligned} \tag{3.24}$$

On the other hand, by Lemma 1.1,

$$\begin{aligned}
\left(|T||T^*|^2|T| + K_3\right)^{\alpha/2} &= \left(|T||T^*|^2|T|\right)^{\alpha/2} + K', \\
\left(|T^*||T|^2|T^*| + K_5\right)^{\alpha/2} &= \left(|T^*||T|^2|T^*|\right)^{\alpha/2} + K'',
\end{aligned} \tag{3.25}$$

with $K', K'' \in \mathcal{C}_{2p/\alpha}(\mathcal{H})$.

Then by (3.21) and (3.24), we obtain

$$\left(|T||T^*|^2|T|\right)^{\alpha/2} + K' \leq |T|^{2\alpha}, \qquad \left(|T^*||T|^2|T^*|\right)^{\alpha/2} + K'' \geq |T^*|^{2\alpha}, \tag{3.26}$$

with $K', K'' \in \mathcal{C}_{2p/\alpha}(\mathcal{H})$. □

Theorem 3.6. *Let $p \geq 1, \alpha \in (0,1]$, and$T \in \mathcal{H}_p^{\alpha}(\mathcal{H})$ such that $D_T^{\alpha} = P + K$ with $P \geq 0, K \in \mathcal{C}_p(\mathcal{H})$, and $D_T^{\alpha} = P_1 - K_1$ with $P_1 \geq 0, K_1 \geq 0$, and $K_1 \in \mathcal{C}_p(\mathcal{H})$. Then if $|T|^{2\alpha} - P \geq 0$, one has $T^2 \in \mathcal{H}_{2p/\alpha}^{\alpha/2}(\mathcal{H})$.*

Proof. Let α, p, and T be as in the hypotheses. We may assume that $T = U|T|$ with U being unitary. Then obviously,

$$\left\{T^2\left(T^2\right)^*\right\}^{\alpha/2} = U\left(|T||T^*|^2|T|\right)^{\alpha/2}U^*, \tag{3.27}$$

$$\left\{\left(T^2\right)^*T^2\right\}^{\alpha/2} = \left(|T|U^*|T|^2U|T|\right)^{\alpha/2} = U^*\left(|T^*||T|^2|T^*|\right)^{\alpha/2}U. \tag{3.28}$$

By Lemma 3.5, there exits $K', K'' \in \mathcal{C}_{2p/\alpha}(\mathcal{H})$ such that

$$\left(|T||T^*|^2|T|\right)^{\alpha/2} + K' \leq |T|^{2\alpha}, \tag{3.29}$$

$$\left(|T^*||T|^2|T^*|\right)^{\alpha/2} + K'' \geq |T^*|^{2\alpha}. \tag{3.30}$$

Multiplying (3.29) by U to the left and by U^* to the right, we obtain

$$U\left(|T||T^*|^2|T|\right)^{\alpha/2}U^* + UK'U^* \leq U|T|^{2\alpha}U^* = |T^*|^{2\alpha}. \tag{3.31}$$

Multiplying (3.30) by U^* to the left and by U to the right, we obtain

$$U^*\left(|T^*||T|^2|T^*|\right)^{\alpha/2}U + U^*K''U \geq U^*|T^*|^{2\alpha}U = |T|^{2\alpha}. \tag{3.32}$$

By (3.27) and (3.31), we have

$$\left\{T^2\left(T^2\right)^*\right\}^{\alpha/2} + UK'U^* \leq |T^*|^{2\alpha}. \tag{3.33}$$

By (3.28) and (3.32), we have

$$\left\{\left(T^2\right)^*T^2\right\}^{\alpha/2} + U^*K''U \geq |T|^{2\alpha}. \tag{3.34}$$

Setting $K_2 = UK'U^* - U^*K''U, K_2 \in \mathcal{C}_{2p/\alpha}(\mathcal{H})$, we have

$$\left\{\left(T^2\right)^*T^2\right\}^{\alpha/2} - \left\{T^2\left(T^2\right)^*\right\}^{\alpha/2} \geq |T|^{2\alpha} - |T^*|^{2\alpha} + K_2. \tag{3.35}$$

Therefore, for $K_3 = K + K_2, K_3 \in \mathcal{C}_{2p/\alpha}(\mathcal{H})$, we have

$$\left\{\left(T^2\right)^*T^2\right\}^{\alpha/2} - \left\{T^2\left(T^2\right)^*\right\}^{\alpha/2} \geq P + K_3. \tag{3.36}$$

Then the proof of Theorem 3.6 is finished. □

Acknowledgment

The authors would like to express their cordial gratitude to the referee for his kind comments.

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